Chap 3 State Variable Models

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Background

- Classic control theory – transfer function approach
  - For linear-time-invariant (LTI) and single-input-single-output (SISO) systems

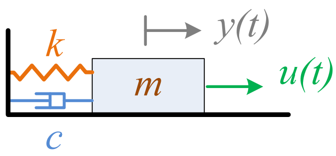
Differential equations (time domain) $\xrightarrow{\mathcal{L}}$ Algebraic equations (s domain) $\xrightarrow{\mathcal{L}^{-1}}$ Solution (time domain)

- Modern control theory – state-space approach
  - Deal with differential equations directly
  - Can be utilized for time-variant, nonlinear, and multiple-input-multiple-output (MIMO) systems
  - 線性控制 為其他進階控制課程的基礎

Five Representations of a System -1

以SMD system為例

$$input = u$$
$$output = y$$



- (1) Differential equation $\quad m\ddot{y} + c\dot{y} + ky = u$

- (2) Transfer function $\quad G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{ms^2 + cs + k}$

  note I.C.s = 0

- (3) Impulse response $\quad Y(s) = G(s)U(s) = \dfrac{1}{ms^2 + cs + k}$

  $u(t) = \delta(t) \xrightarrow{\mathcal{L}} U(s) = 1$

Five Representations of a System -2

- (4) State-space

$$output = y = x_1$$
$$\dot{y} = x_2 = \dot{x_1}$$
$$\ddot{y} = \dot{x_2} = \frac{-c}{m}\dot{y} + \frac{-k}{m}y + \frac{1}{m}u = \frac{-c}{m}x_2 + \frac{-k}{m}x_1 + \frac{1}{m}u$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2x1} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A\ 2x2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2x1} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B\ 2x1} u_{1x1}$$

$$y_{1x1} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C\ 1x2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2x1} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D\ 1x1} u_{1x1}$$

Note:
A 3rd-order system
2 inputs
2 outputs

x: 3x1
u: 2x1
y: 2x1

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

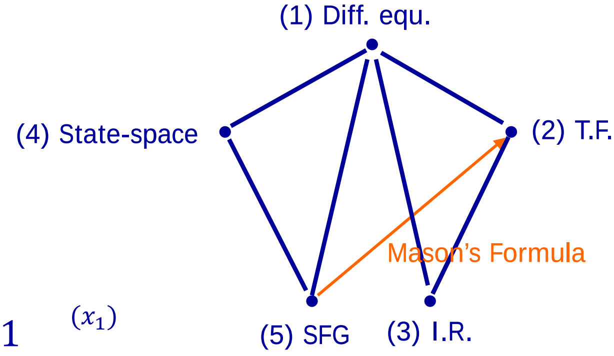
A: 3x3
B: 3x2
C: 2x3
D: 2x2

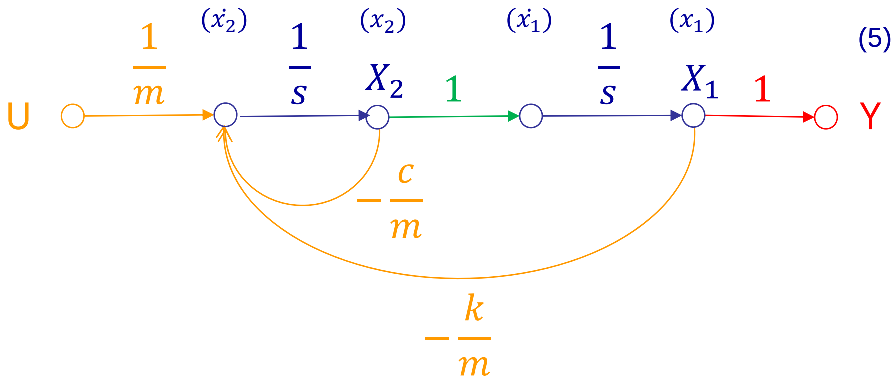
$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{5x5}$$

Five Representations of a System -3

- (5) Signal-flow graph

$$y = x_1$$
$$\dot{y} = x_2 = \dot{x}_1$$
$$\ddot{y} = \frac{-c}{m}\dot{y} + \frac{-k}{m}y + \frac{1}{m}u$$
$$= \frac{-c}{m}x_2 + \frac{-k}{m}x_1 + \frac{1}{m}u$$

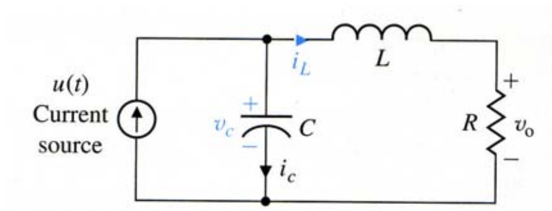




Five Representations of a System -4

⇨ 以RLC circuit為例

$$input = u$$
$$output = v_0$$



- (1) Differential equation

$$i_c = C\frac{dv_c}{dt} = u(t) - i_L \qquad \cdots\cdots(1)$$

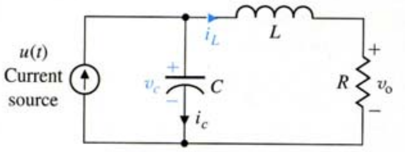
$$L\frac{di_L}{dt} = v_c - i_L R \qquad \cdots\cdots(2)$$

$$v_o = i_L R \qquad \cdots\cdots(3)$$

Five Representations of a System -5

- (2) State-space

Set $x_1 = v_c \quad x_2 = i_L$



$$C\frac{dv_c}{dt} = u(t) - i_L$$

$$L\frac{di_L}{dt} = v_c - i_L R$$

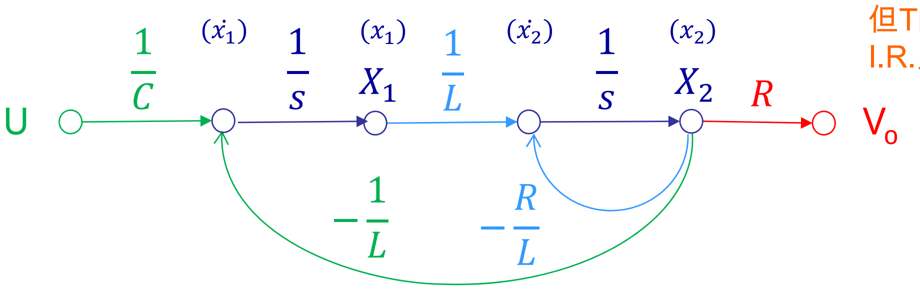
$$v_o = i_L R$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & R \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

針對一個系統，
選擇不同state，就會有不同的(A,B,C,D)和SFG。
但Transfer function和I.R.則為唯一表達

- (3) Signal-flow graph



Five Representations of a System -6

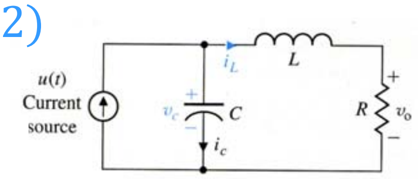
- (4) Transfer function

From (1) $u(t) = i_c + i_L = C\frac{dv_c}{dt} + i_L$

$$= C\frac{d}{dt}\left(L\frac{di_L}{dt} + i_L R\right) + i_L \quad (2)$$

$$= CL\ddot{i}_L + CR\dot{i}_L + i_L$$

$$= \frac{CL}{R}\ddot{v}_o + C\dot{v}_o + \frac{v_o}{R} \quad (3)$$

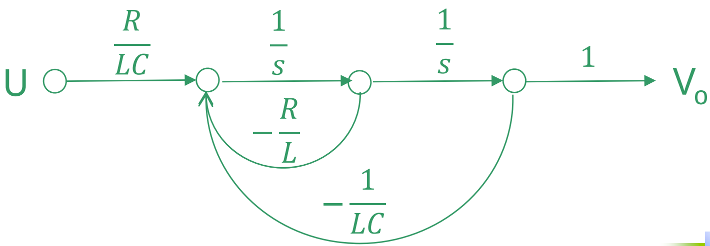


$$\Rightarrow G = \frac{V_o}{U} = \frac{1}{\frac{LC}{R}s^2 + Cs + \frac{1}{R}}$$

- (5) Impulse response

$$V_0 = GU = G$$

仿SMD example中
state架構所得出之SFG



Solving Differential equations -1

- Single variable

$$\dot{x} = ax + bu \xrightarrow{\mathcal{L}} sX(s) - x(0) = aX(s) + bU(s)$$

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

Note: $\mathcal{L}^{-1}[F(s)G(s)] = f(t) * g(t)$

Solving Differential equations -2

- Multiple variables

$$\underset{\text{nx1}}{\dot{\boldsymbol{x}}} = \underset{\text{nxn nx1}}{A\boldsymbol{x}} + \underset{\text{nxm mx1}}{B\boldsymbol{u}} \xrightarrow{\mathcal{L}} sX(s) - \boldsymbol{x}(0) = AX(s) + BU(s)$$

$$(sI - A)X(s) = \boldsymbol{x}(0) + BU(s)$$

$$X(s) = (sI - A)^{-1}\boldsymbol{x}(0) + (sI - A)^{-1}BU(s)$$

$$= \Phi(s)\boldsymbol{x}(0) + \Phi(s)BU(s)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$x(t) = \Phi(t)\boldsymbol{x}(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

$$= e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

State transition matrix

$$\Phi(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \Phi(s)$$

$$\Phi(t) = e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$$

$$\Phi(s) = (sI - A)^{-1}$$

State-space to Transfer Function -1

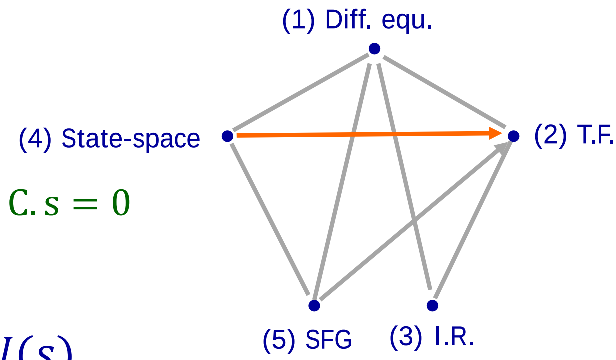
- $\dot{x} = Ax + Bu$

  $y = Cx + Du$

由前一頁公式

$$X(s) = (sI - A)^{-1}BU(s) \quad \text{note I.C.s} = 0$$

$$\begin{aligned} Y(s) &= CX(s) + DU(s) \\ &= C(sI - A)^{-1}BU(s) + DU(s) \\ &= [C(sI - A)^{-1}B + D]U(s) \\ &= G(s)U(s) \end{aligned}$$



⇨ $G(s) = C(sI - A)^{-1}B + D$

State-space to Transfer Function -2

- Revisit the RLC circuit

$$A = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$C = [0 \quad R] \qquad D = 0$$

Note: $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$M^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$MM^{-1} = I$$

$$sI - A = \begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s + \frac{R}{L} \end{bmatrix}$$

$$\Phi(s) = (sI - A)^{-1} = \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}$$

$$G(s) = [0 \quad R]\Phi(s)\begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} = \frac{\frac{R}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

和p8結果相同

Evaluation of the State Transition Matrix -1

- Revisit the RLC circuit

$$R = 3 \quad L = 1 \quad C = \frac{1}{2} \qquad \therefore A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$$

$$\Phi(s) = (sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}$$

$$\Phi_{11} = \frac{(s+3)}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$

同理可得Φ_{12} Φ_{21} Φ_{22}

$$\Phi(t) = \mathcal{L}^{-1}[\Phi(s)] = \begin{bmatrix} 2e^{-t} - e^{-2t} & -2e^{-t} + 2e^{-2t} \\ e^{-t} - e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Evaluation of the State Transition Matrix -2

- Revisit the RLC circuit

Another method: $eigen\ decomposition$

eigenvectors

$AV = V\Lambda$

eigenvalues

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A = V\Lambda V^{-1}$$

$$e^{At} = Ve^{\Lambda t}V^{-1}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & -2e^{-t} + 2e^{-2t} \\ e^{-t} - e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

和前頁p13結果相同

Transfer Function to State-space -1

- Ex: $G(s) = \dfrac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$

$$= \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \frac{Z(s)}{Z(s)} = \frac{Y(s)}{U(s)}$$

$$Y(s) = (b_3 s^3 + b_2 s^2 + b_1 s + b_0)Z(s)$$

$$y(t) = b_3 z^{(3)} + b_2 \ddot{z} + b_1 \dot{z} + b_0 z$$

$$U(s) = (s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0)Z(s)$$

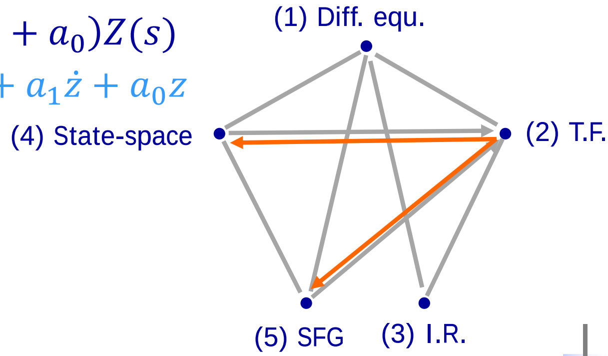
$$u(t) = z^{(4)} + a_3 z^{(3)} + a_2 \ddot{z} + a_1 \dot{z} + a_0 z$$

Assign $x_1 = z$

$x_2 = \dot{z} = \dot{x_1}$

$x_3 = \ddot{z} = \dot{x_2}$

$x_4 = z^{(4)} = \dot{x_3}$

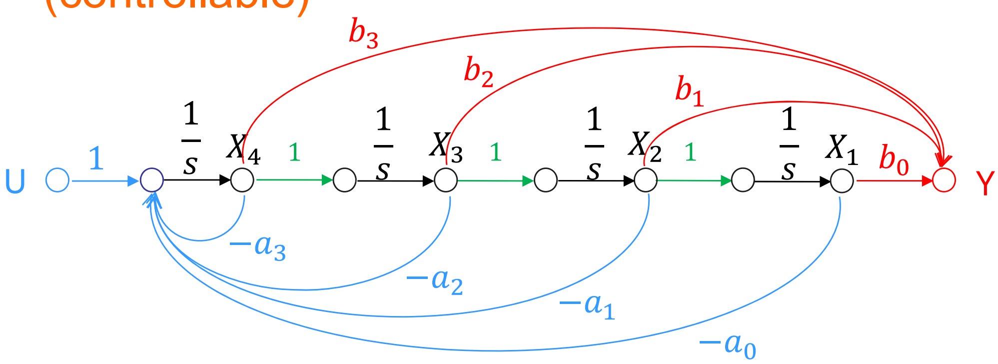


Transfer Function to State-space -2

- $$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

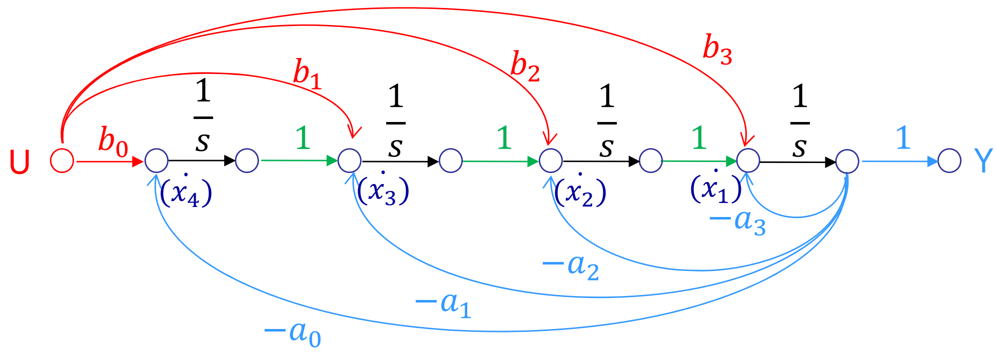
Phase variable canonical form
(controllable)



Transfer Function to State-space -3

- $$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-a_3 & 1 & 0 & 0\\-a_2 & 0 & 1 & 0\\-a_1 & 0 & 0 & 1\\-a_0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}+\begin{bmatrix}b_3\\b_2\\b_1\\b_0\end{bmatrix}u$$
$$y=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

Input forward canonical form
(observable)



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- Questions?