Chap 9 Stability in the Frequency Domain

林沛群
國立台灣大學
機械工程學系

Introduction

- Roodmap
  - Chapter 6 : Routh-Hurwitz Criterion

    Check stability of the system by examining $\Delta(s)$

    Introduce idea of relation stability
  - Chapter 7 : Root Locus

    Investigate loci of the system poles as the system parameter changes
  - Chapter 8 : Polar plot & Bode plot

    Introduce frequency response of the system
  - This Chapter :

    Investigate stability of the system in the frequency domain

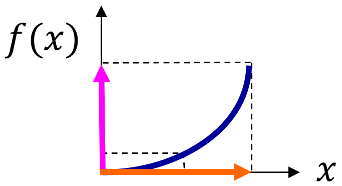
Mapping Contours in the s-plane

- Real-number functions

$x, f(x) \in R$

Ex: $f(x) = x^3$

$f(x) : x \rightarrow f(x)$



- Contour map
  - A contour or trajectory in one plane mapped or translated into another plane by a relation $F(s)$

$$\begin{array}{ccc} F(s) : & s & \rightarrow F(s) \\ & \downarrow & \downarrow \\ & \sigma + j\omega & u + jv \end{array}$$

Mapping Contours in the s-plane

- Example: $F(s) = 2s + 1$

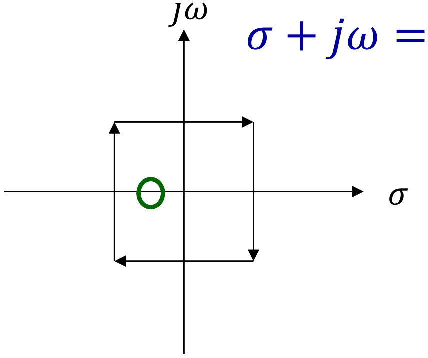
$$u + jv = F(s) = 2s + 1 = 2(\sigma + j\omega) + 1$$

$$= (2\sigma + 1) + j(2\omega)$$

scaling shifting

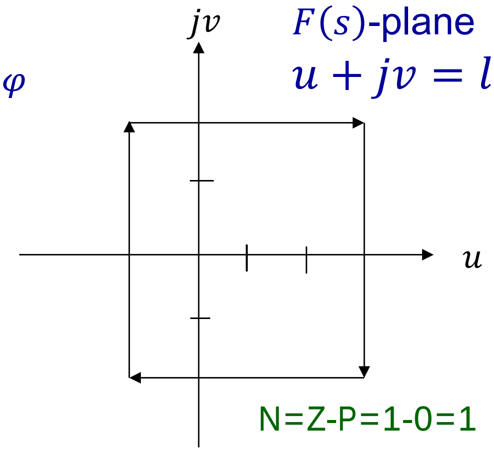
s-plane

$\sigma + j\omega = re^{j\varphi}$



$F(s)$-plane

$u + jv = le^{j\theta}$



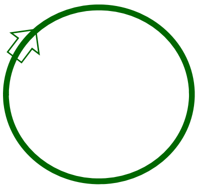
N=Z-P=1-0=1

Mapping Contours in the s-plane

- $F(s): s \rightarrow F(s)$, a conformal map, which retains the angles of the s-plane contour on the F(s) plane

- By convention, the area within a contour to the right of the traversal of the contour is considered to be the area enclosed by the contour

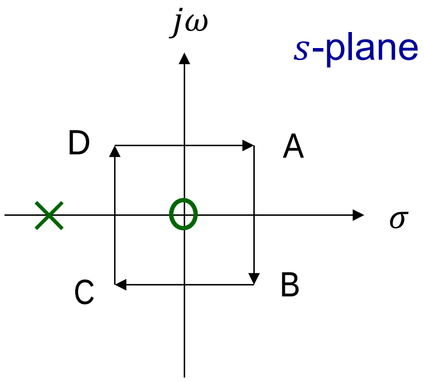
  Clockwise : "+"

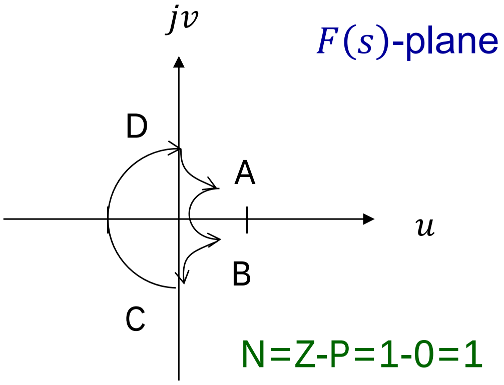


Mapping Contours in the s-plane

- Example: $F(s) = \dfrac{s}{s+2}$

A	$\longrightarrow$	B	$\longrightarrow$	C	$\longrightarrow$	D	$\longrightarrow$
$1+j1$	1	$1-j1$	$-j1$	$-1-j1$	-1	$-1+j1$	$-j1$
$\frac{4+2j}{10}$	$\frac{1}{3}$	$\frac{4-2j}{10}$	$\frac{1-2j}{5}$	$-j$	-1	$+j$	$\frac{1+2j}{5}$





N=Z-P=1-0=1

Cauchy’s Theorem -1

- Suppose : $F(s)$ = c.l. characteristic equ.

  question : How to judge the stability of the closed-loop system given the open-loop transfer function $L(s)$?

$$\Delta(s) = F(s) = 1 + L(s)$$

answer : Cauchy’s Theorem & Nyquist Criterion

Note : in Chapter 7　　課本符號不統一

$$\Delta(s) = 1 + KG(s) = 1 + F(s)$$

Cauchy’s Theorem -2

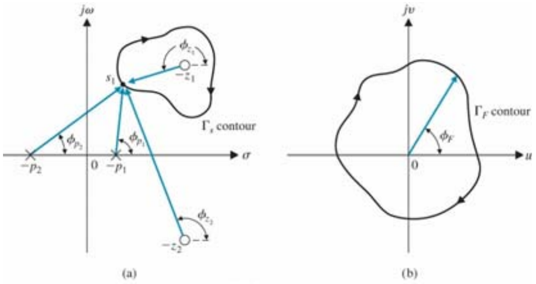
- If a contour Γ_s in the s-plane

  (1) encircles Z zeros and P poles of $F(s)$,

  (2) does not pass through any poles or zeros of $F(s)$, and

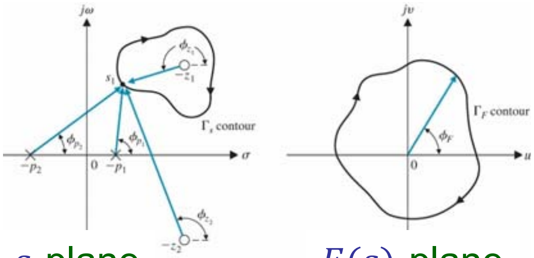
  (3) the traversal is in the clockwise direction along the contour,

  the corresponding contour Γ_F in the $F(s)$-plane encircles the origin of the $F(s)$-plane $N = Z - P$ times in the clockwise direction



Cauchy's Theorem -3

- Example: $F(s) = \dfrac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$



s-plane $F(s)$-plane

$$F(s) = |F(s)|\angle F(s)$$
$$= \frac{|s+z_1||s+z_2|}{|s+p_1||s+p_2|}(\angle(s+z_1)+\angle(s+z_2)-\angle(s+p_1)-\angle(s+p_2))$$
$$= |F(s)|(\angle\phi_{z_1}+\angle\phi_{z_2}-\angle\phi_{p_1}-\angle\phi_{p_2})$$

As s traverses the contour Γ_s (a full rotation)

$\phi_{z_2}, \phi_{p_1}, \phi_{p_2}: \quad net\ angle\ change = 0$

$\phi_{z_1}: \quad net\ angle\ change = 360°$

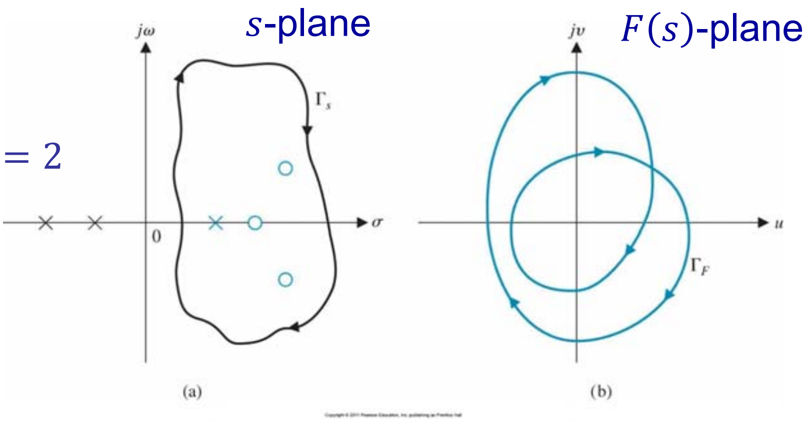
If $F(s)$ has Z zeros and P poles, $\phi_z = 2\pi(Z)$ $\phi_p = 2\pi(P)$

$$\phi_F = \phi_Z - \phi_P$$
$$\Rightarrow \quad N = Z - P$$

Cauchy's Theorem -4

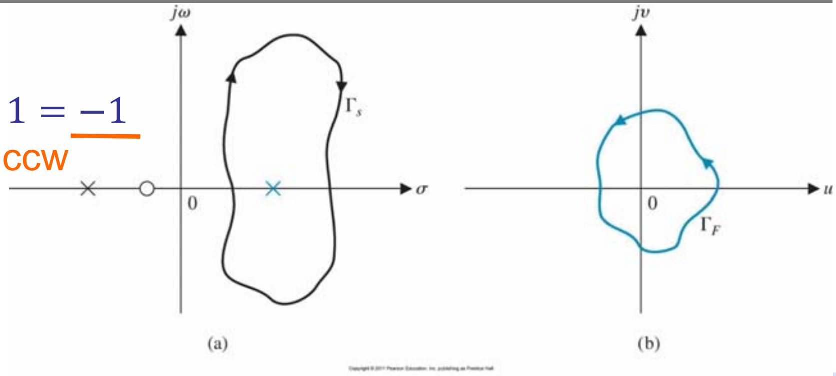
- Examples:

$N = Z - P = 3 - 1 = 2$



$N = Z - P = 0 - 1 = \underline{-1}$

ccw



Nyquist Criterion -1

- Logic

$L(s) = \frac{N(s)}{D(s)}$: Loop T.F., known

$T(s) = \frac{\cdots}{\Delta(s)}$: Closed-loop T.F.

$$\Delta(s) = F(s) = \frac{k\prod_{i=1}^{N}(s+s_i)}{\prod_{k=1}^{M}(s+s_k)} = 1 + L(s) = 1 + \frac{N(s)}{D(s)} = \frac{D(s)+N(s)}{D(s)}$$

poles of $F(s)$ = poles of $L(s)$ = roots of $D(s)$ known

zeros of $F(s)$ = $\underline{\text{poles of } T(s)}$ = roots of $D(s)+N(s)$ unknown

Determining stability of the system

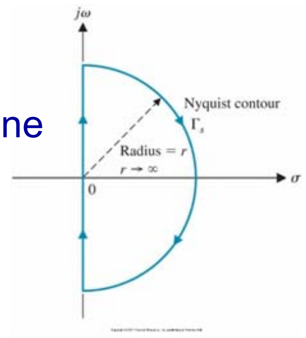
Nyquist Criterion -2

A stable c.l. system ⇒ all “poles” of $T(s)$ in LHP

⇒ all “zeros” of $F(s)$ in LHP

⬇

Choose a contour Γ_s encloses the entire RHP in s -plane



⬇

Cauchy’s Theorem $Z = N + P$

knowing N by Γ_F’s encirclement of (0,0) in $F(s)$-plane

or Γ_L’s encirclement of (-1,0) in $L(s)$-plane

$F(s) = 1 + L(s)$

knowing P by $L(s)$ its poles

⇒ Know Z zeros of $F(s)$ in RHP

⇒ Know Z poles of $T(s)$ in RHP

Nyquist Criterion -3

- A feedback system is stable if and only if the contour Γ_L in the $L(s)$-plane does NOT encircle the (-1,0) point when the number of poles of $L(s)$ in the right-hand s-plane is zero ($P = 0$)

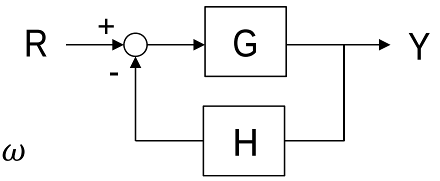
$$Z = N + P = 0 + 0 = 0$$

- A feedback control system is stable if and only if, for the contour Γ_L, the number of counterclockwise encirclement of the (-1,0) point is equal to the number of poles of $L(s)$ with positive real parts

$$Z = N + P = 0$$

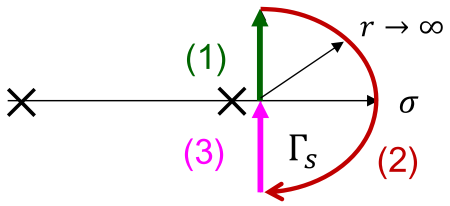
Example 1 -1

- $L(s) = GH(s) = \frac{K}{(\tau_1 s+1)(\tau_2 s+1)} \quad K > 0$



$$Assume\ \tau_1 = 1 \quad \tau_2 = \frac{1}{10}$$

$$(1)\ \omega = 0 \ \rightarrow \ \omega = +\infty$$



$$GH(j\omega) = GH(s)\Big|_{s=j\omega}$$

$$= \frac{10K(10-\omega^2)}{(10-\omega^2)^2-(11\omega^2)^2} + j\frac{-10K(11\omega)}{(10-\omega^2)^2-(11\omega^2)^2}$$

Cross Im-axis at $Im|_{\omega=\sqrt{10}} = 0.287K$ when $\omega = \sqrt{10}$

$$|GH| = 10K\sqrt{\frac{1}{(10-\omega^2)^2+(11\omega^2)^2}} \qquad \phi = -\tan^{-1}(\frac{11\omega}{10-\omega^2})$$

"4 quadrants"

Example 1 -2

ω	0	1	$\sqrt{10}$	10	100	∞
$\|GH\|$	100	70.7	28.74	6.8	0.1	0
ϕ	0	-50.7	-90	-129.3	-173.7	-180

P.S. The “polar plot” in Chapter 8

$$(2)\ \omega = +\infty \ \rightarrow \ \omega = -\infty$$

$$s = re^{j\phi} : \begin{cases} \phi = 90^\circ \rightarrow -90^\circ \quad cw \\ r \rightarrow \infty \end{cases}$$

$$L(s) = le^{j\theta} = \lim_{r\rightarrow\infty} GH(s)\Big|_{s=re^{j\phi}} = \lim_{r\rightarrow\infty}\left|\frac{K}{\tau_1\tau_2 r^2}\right| e^{-j2\phi}$$

$$\begin{cases} \theta = -180^\circ \rightarrow +180^\circ \quad ccw \\ l \rightarrow 0 \end{cases}$$

P. S. → 0 when the denominator has higher order than the numerator

---

Example 1 -3

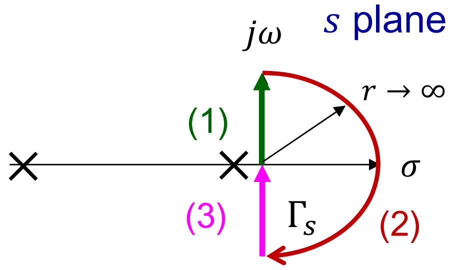
$$(3)\ \omega = -\infty \ \rightarrow \ \omega = 0$$

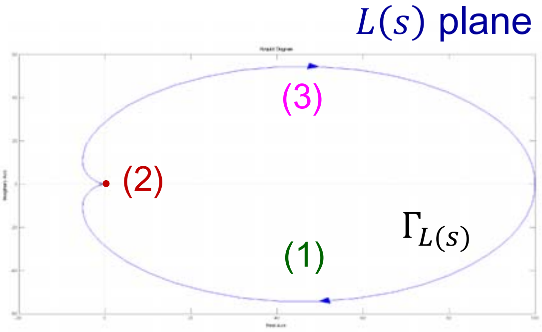
$$L(s):\quad GH(s)\Big|_{s=-j\omega} = GH(-j\omega) = complex\ conjugate\ of\ GH(j\omega)$$

$$\omega = +\infty \ \rightarrow \ \omega = 0^+$$

⇨ $mirrored\ of\ (1)\ w.r.t.\ Re - axis$
$and\ change\ arrow\ direction$

Thus,



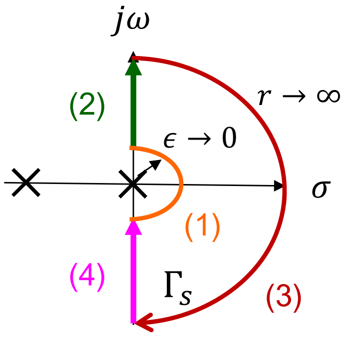


$$Z = N + P = 0 + 0 = 0 \ \ stable, \quad K \ \ dosen't\ matter$$

Example 2 -1

- $L(s) = GH(s) = \frac{K}{s(\tau s+1)} \quad K > 0, \ \ \tau > 0$

$pole\ at\ origin \rightarrow \Gamma_s\ needs\ detour$



(1) $\omega = 0^- \rightarrow \ \ \omega = 0^+$

$$s = re^{j\phi}: \begin{cases} \phi = -90^\circ \rightarrow 90^\circ \quad ccw \ \circlearrowleft \\ r = \epsilon \rightarrow 0 \end{cases}$$

$$L(s) = le^{j\theta} = \lim_{\epsilon \to 0} GH(s) \approx \lim_{\epsilon \to 0} \left(\frac{K}{\epsilon e^{j\phi}}\right) = \lim_{\epsilon \to 0} \left(\frac{K}{\epsilon}\right) e^{-j\phi}$$

$$\begin{cases} \theta = 90^\circ \rightarrow -90^\circ \quad cw \ \circlearrowright \\ l \rightarrow \infty \end{cases}$$

P.S. an infinite half circle

Example 2 -2

(2) $\omega = 0^+ \ \rightarrow \ \omega = +\infty$

The same as the “polar plot” example shown in Chap. 8

$$GH(j\omega) = \frac{K}{j\omega(j\omega\tau + 1)} = \frac{K}{-\omega^2\tau + j\omega} = \frac{-K\omega^2\tau}{\omega^4\tau^2 + \omega^2} + \frac{-j\omega K}{\omega^4\tau^2 + \omega^2}$$

$\pm\omega \rightarrow \infty$: symmetry w.r.t. Re-axis

	R	**X**
$\omega = 0$	$-K\tau$	$-\infty$
$\omega = \frac{1}{2}$	$-\frac{K\tau}{2}$	$-\frac{K\tau}{2}$
$\omega = \infty$	0	0

	$\boldsymbol{G}$	$\boldsymbol{\phi}$
$\omega = 0$	∞	-90°
$\omega = \frac{1}{2}$	$\frac{K\tau}{\sqrt{2}}$	-135°
$\omega = \infty$	0	180°

$$|GH| = \frac{K}{(\omega^4\tau^2 + \omega^2)^{\frac{1}{2}}}$$

“4 quadrants”

$$\phi(\omega) = -\tan^{-1}\left(\frac{1}{-\omega\tau}\right) \ or = -\frac{\pi}{2} - \tan^{-1}\omega\tau$$

Example 2 -3

(3) $\omega = +\infty \;\rightarrow\; \omega = -\infty$

$$s = re^{j\phi} : \begin{cases} \phi = 90^\circ \rightarrow -90^\circ \quad cw \;\circlearrowright \\ \quad r \rightarrow \infty \end{cases}$$

$$L(s) = le^{j\theta} = \left. \lim_{r\to\infty} GH(s) \right|_{s=re^{j\phi}} = \lim_{r\to\infty} \left| \frac{K}{\tau r^2} \right| e^{-j2\phi}$$

$$\begin{cases} \theta = -180^\circ \rightarrow +180^\circ \quad ccw \;\circlearrowleft \\ \quad l \rightarrow 0 \end{cases}$$

(4) $\omega = -\infty \;\rightarrow\; \omega = 0^-$

$$L(s):\quad \left. GH(s) \right|_{s=-j\omega} = GH(-j\omega) = complex\ conjugate\ of\ GH(j\omega)$$

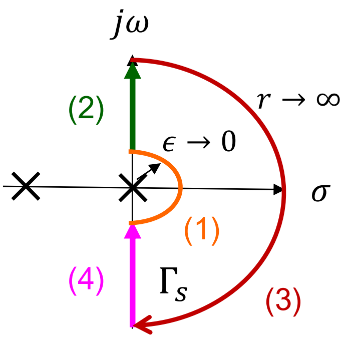
$$\omega = +\infty \;\rightarrow\; \omega = 0^+$$

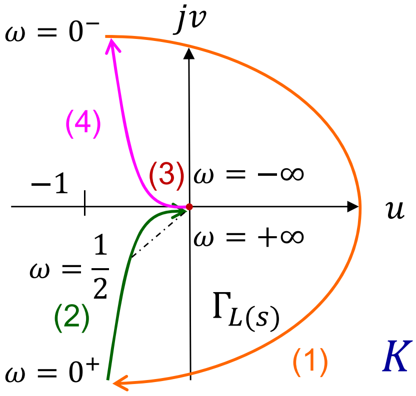
⇨ $mirrored\ of\ (1)\ w.r.t.\ Re - axis$

$and\ change\ arrow\ direction$

Example 2 -4

Thus,





$$\mathrm{Z = N + P}$$
$$= 0 + 0 = 0$$

⇨ $stable$

$K\ \ dosen't\ matter$

Note: Routh-Hurwitz & Root locus methods

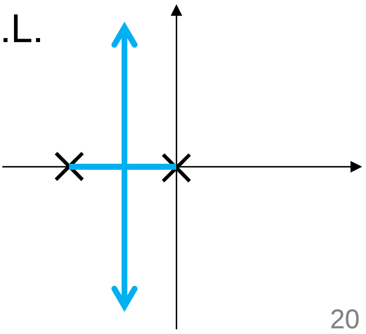
$$L(s) = GH(s) = \frac{K}{s(\tau s + 1)} \quad K > 0$$

R.H.

$$\tau s^2 + s + K = 0$$

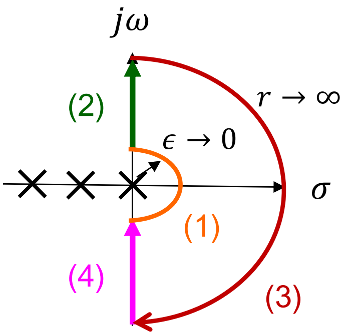
Stable as long as $K > 0$

R.L.



Example 3 -1

- ☐ $GH(s) = \frac{K}{s(\tau_1 s+1)(\tau_2 s+1)} \quad K > 0, \tau_1 > 0, \tau_2 > 0$



$$\omega:\quad \underbrace{0^- \xrightarrow{(1)} 0^+}_{infinite\ half\ cycle}\ \underbrace{\xrightarrow{(2)} \infty}\ \underbrace{\xrightarrow{(3)} -\infty}_{at\ origin}\ \underbrace{\xrightarrow{(4)} 0^-}$$

$conjugate\ (symmetry\ w.r.t.\ Re - axis)$

(1) $\omega = 0^- \to \ \omega = 0^+$

$$s = re^{j\phi} : \begin{cases} \phi = -90° \to 90° \quad ccw \\ \qquad r = \epsilon \to 0 \end{cases}$$

$$L(s) = le^{j\theta} = \lim_{\epsilon \to 0} \frac{K}{\epsilon e^{j\phi}(\tau_1 \epsilon e^{j\phi} + 1)(\tau_2 \epsilon e^{j\phi} + 1)} \approx \lim_{\epsilon \to 0}\left(\frac{K}{\epsilon e^{j\phi}}\right)$$

$$= \lim_{\epsilon \to 0}\left(\frac{K}{\epsilon}\right) e^{-j\phi}$$

$$\begin{cases} \theta = 90° \to -90° \quad cw \\ \qquad l \to \infty \end{cases}$$

P.S. an infinite half circle

Example 3 -2

(2) $\omega = 0^+ \ \to \ \omega = +\infty$

$$GH(j\omega) = \frac{-K(\tau_1 + \tau_2) - jK(\frac{1}{\omega})(1 - \omega^2 \tau_1 \tau_2)}{1 + \omega^2(\tau_1^2 + \tau_2^2) + \omega^4 \tau_1^2 \tau_2^2}$$

$$when\ \omega^2 = \frac{1}{\tau_1 \tau_2}, \qquad X = v = \frac{K\frac{1}{\omega}(1 - \omega^2 \tau_1 \tau_2)}{\sim} = 0$$

$across\ Re - axis\ at$

$$R = u = \left.\frac{-K(\tau_1 + \tau_2)}{1 + \omega^2(\tau_1^2 + \tau_2^2) + \omega^4 \tau_1 \tau_2}\right|_{\omega^2 = \frac{1}{\tau_1 \tau_2}} = \frac{-K\tau_1 \tau_2}{\tau_1 + \tau_2}$$

Example 3 -3

(3) $\omega = +\infty \;\rightarrow\; \omega = -\infty$

$$s = re^{j\phi} : \begin{cases} \phi = 90^\circ \rightarrow -90^\circ \quad cw \\ \quad r \rightarrow \infty \end{cases}$$

$$L(s) = le^{j\theta} = \lim_{r\to\infty} \frac{K}{re^{j\phi}\underbrace{(\tau_1 re^{j\phi}+1)}_{\sim\tau_1 re^{j\phi}}\underbrace{(\tau_2 re^{j\phi}+1)}_{\sim\tau_2 re^{j\phi}}} = \Big|_{s=re^{j\phi}}$$

$$\approx \lim_{r\to\infty} \left|\frac{K}{\tau_1\tau_2 r^3}\right| e^{-j3\phi}$$

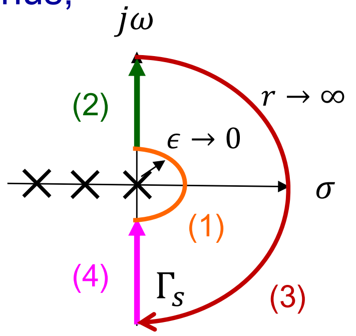
$$\begin{cases} \theta = -270^\circ \rightarrow +270^\circ \quad ccw \\ \quad l \rightarrow 0 \end{cases}$$

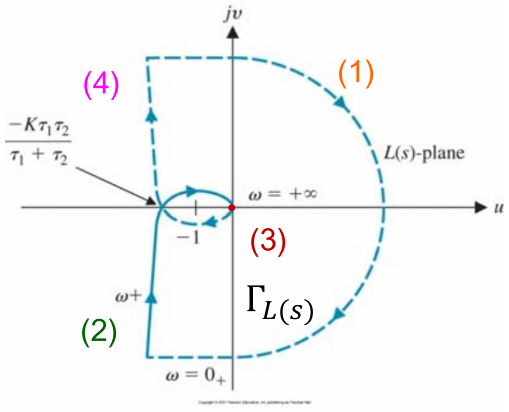
(4) $\omega = -\infty \;\rightarrow\; \omega = 0^-$

mirrored of (1) w.r.t. Re − axis and change arrow direction

Example 3 -4

Thus,





when $\frac{-K\tau_1\tau_2}{\tau_1+\tau_2} > -1,\; N = 0, \qquad Z = N + P = 0 + 0 = 0$ *stable*

$$K < \frac{\tau_1+\tau_2}{\tau_1\tau_2}$$

when $\frac{-K\tau_1\tau_2}{\tau_1+\tau_2} < -1, N = 2, \qquad Z = 2 + 0 = 2$ *unstable*

"2 RHP poles"

Example 3 -5

Note: Routh-Hurwitz & Root locus methods

$$GH(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

R.H.

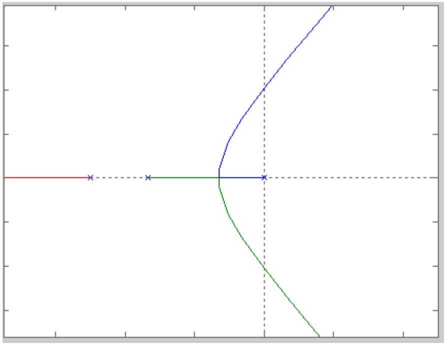
$$\tau_1\tau_2 s^3 + (\tau_1 + \tau_2)s^2 + s + K = 0$$

① $K > 0$

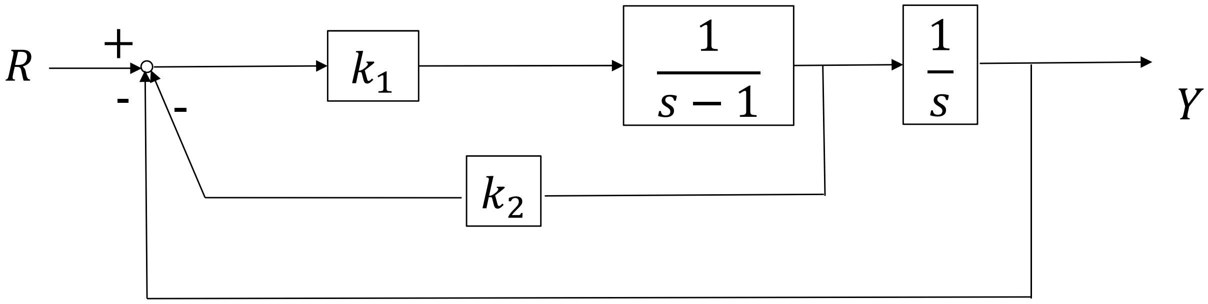
② $(\tau_1 + \tau_2) > \tau_1\tau_2 K$

$$\rightarrow K < \frac{\tau_1+\tau_2}{\tau_1\tau_2}$$

R.L.



Example 4 -1



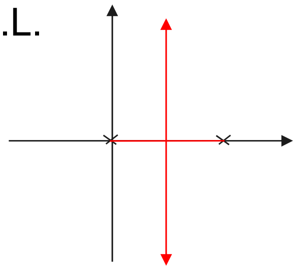
- $k_2 = 0,\ \ without\ derivative\ feedback$

$$GH(s) = \frac{k_1}{s(s-1)}$$

$P = 1,$

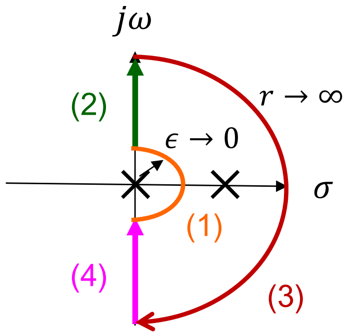
$stable\ system\ (Z = 0):\ N = -P = -1$

R.L.



$\Delta(s) = s^2 - s + k_1$

Example 4 -2



$$\omega:\quad \underbrace{\overset{(1)}{0^- \to 0^+}}_{infinite\ half\ cycle}\ \underbrace{\overset{(2)}{\to \infty}}\ \underbrace{\overset{(3)}{\to -\infty}}_{at\ origin}\ \underbrace{\overset{(4)}{\to 0^-}}$$

$conjugate\ (symmetry\ w.r.t.\ Re - axis)$

$(1)\ \omega = 0^- \to \ \ \omega = 0^+$

$$s = re^{j\phi}:\ \begin{cases} \phi = -90^\circ \to 90^\circ \quad ccw \\ \quad r = \epsilon \to 0 \end{cases}$$

$$L(s) = le^{j\theta} = \lim_{\epsilon \to 0} \frac{k_1}{\epsilon e^{j\phi}(\epsilon e^{j\phi} - 1)} \approx \lim_{\epsilon \to 0}\left(\frac{k_1}{-\epsilon e^{j\phi}}\right) = \lim_{\epsilon \to 0}\left(\frac{k_1}{\epsilon}\right) e^{j(-180-\phi)}$$

$$\begin{cases} \theta = -90^\circ \to -270^\circ \quad cw \\ \quad l \to \infty \end{cases}$$

P.S. an infinite half circle

Example 4 -3

$(2)\ \omega = 0^+ \ \to \ \omega = +\infty$

$$GH(j\omega) = \frac{k_1}{j\omega(j\omega - 1)} = \frac{-k_1\omega^2\tau + jk_1\omega}{\omega^4\tau^2 + \omega^2}$$

$(3)\ \omega = +\infty \ \to \ \omega = -\infty$

$$s = re^{j\phi}:\ \begin{cases} \phi = 90^\circ \to -90^\circ \quad cw \\ \quad r \to \infty \end{cases}$$

$$L(s) = le^{j\theta} = \left|\lim_{r \to \infty} \frac{k_1}{re^{j\phi}\underbrace{(re^{j\phi} - 1)}_{\sim re^{j\phi}}} = \right|_{s = re^{j\phi}} \approx \lim_{r \to \infty}\left|\frac{K}{r^2}\right| e^{-j2\phi}$$

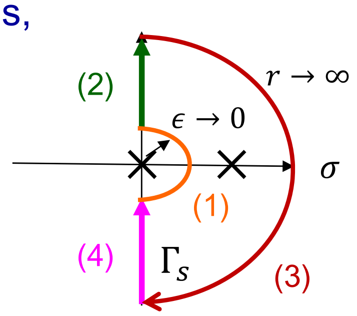
$$\begin{cases} \theta = -180^\circ \to +180^\circ \quad ccw \\ \quad l \to 0 \end{cases}$$

$(4)\ \omega = -\infty \ \to \ \omega = 0^-$

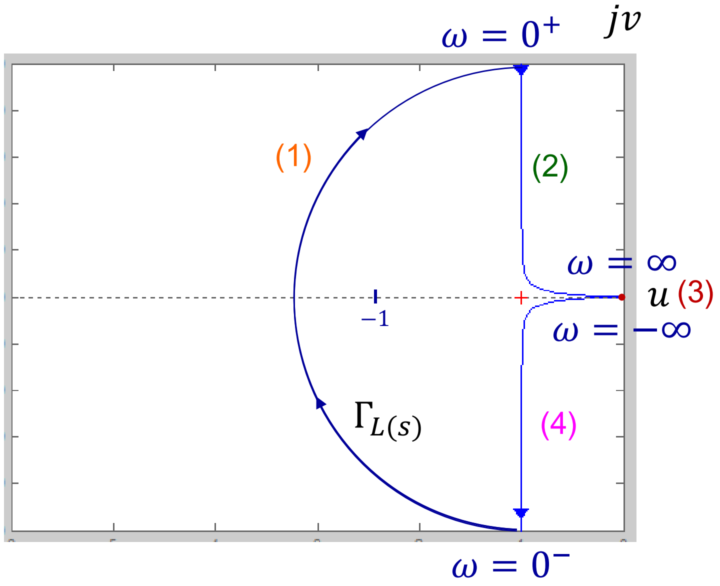
$mirrored\ of\ (1)\ w.r.t.\ Re - axis\ and\ change\ arrow\ direction$

Example 4 - 4

Thus,



	$\frac{\lvert GM\rvert}{k_1}$	$\angle GM$
$j0^-$	∞	-90^o
$j0^+$	∞	90^o
j	$\frac{1}{\sqrt{2}}$	135^o
$+j\infty$	0	180^o
$-j\infty$	0	-180^o

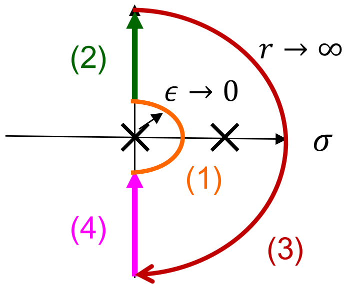


$\therefore N = 1 \;\; Z = 2, \qquad unstable$

Example 4 -5

- $with\ k_2$

$$GH(s) = \frac{k_1(1+k_2 s)}{s(s-1)}$$



(1) $\omega = 0^- \to \;\; \omega = 0^+$

$$s = re^{j\phi} : \begin{cases} \phi = -90° \to 90° \quad ccw \circlearrowleft \\ r = \epsilon \to 0 \end{cases}$$

$$L(s) = le^{j\theta} = \lim_{\epsilon \to 0} \frac{k_1(1+k_2\epsilon e^{j\phi})}{\epsilon e^{j\phi}(\epsilon e^{j\phi}-1)} \approx \lim_{\epsilon \to 0}\left(\frac{k_1}{-\epsilon e^{j\phi}}\right) = \lim_{\epsilon \to 0}\left(\frac{k_1}{\epsilon}\right) e^{j(-180-\phi)}$$

$$\begin{cases} \theta = -90° \to -270° \quad cw \circlearrowright \\ l \to \infty \end{cases}$$

Example 4 -6

(2) $\omega = 0^+ \ \rightarrow \ \omega = +\infty$

$$GH(j\omega) = \frac{k_1(1+k_2 j\omega)}{-\omega^2 - j\omega} = -\frac{k_1(\omega^2+\omega^2 k_2)}{\omega^2+\omega^4} + \frac{j(\omega - k_2\omega^3)k_1}{\omega^2+\omega^4}$$

$$\omega - k_2\omega^3 = 0 \quad across\ Re - axis$$

$$u|_{\omega^2=\frac{1}{k_2}} = -\frac{k_1(\omega^2+\omega^2k_2)}{\omega^2+\omega^4}\Big|_{\omega^2=\frac{1}{k_2}} = -k_1k_2$$

(3) $\omega = +\infty \ \rightarrow \ \omega = -\infty$

$$s = re^{j\phi} : \begin{cases} \phi = 90° \ \rightarrow -90° \quad cw \\ \qquad r \rightarrow \infty \end{cases}$$

$$L(s) = le^{j\theta} = \left| \lim_{r\rightarrow\infty} \frac{k_1(1+k_2re^{j\phi})}{re^{j\phi}(re^{j\phi}-1)} = \right|_{s=re^{j\phi}} \approx \lim_{r\rightarrow\infty} \left|\frac{k_1k_2}{r}\right| e^{-j\phi}$$

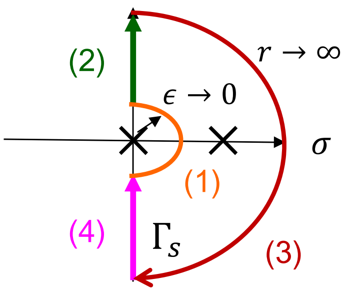
$$\begin{cases} \theta = -90° \ \rightarrow +90° \quad ccw \\ \qquad l \rightarrow 0 \end{cases}$$

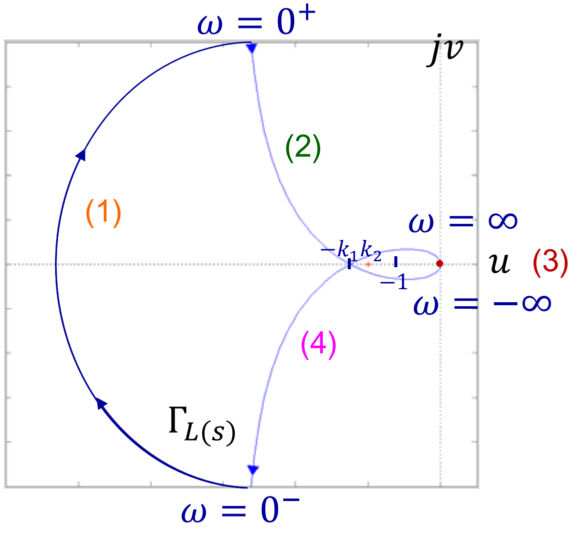
Example 4 -7

(4) $\omega = -\infty \ \rightarrow \ \omega = 0^-$

$$mirrored\ of\ (1)\ w.r.t.\ Re - axis\ and\ change\ arrow\ direction$$

Thus,





$when\ k_1k_2 > 1$

$\Gamma_{F(s)}\ encircles\ -1\ once\ in\ ccw\ direction$

$N = -1, \qquad Z = N + P = -1 + 1 = 0,$

$stable$

$$\Delta(s) = s^2 - s + k_1k_2s + k_1$$
$$= s^2 + (k_1k_2 - 1)s + k_1 = 0$$

$k_1k_2 - 1 > 0 \qquad k_1 > 0$

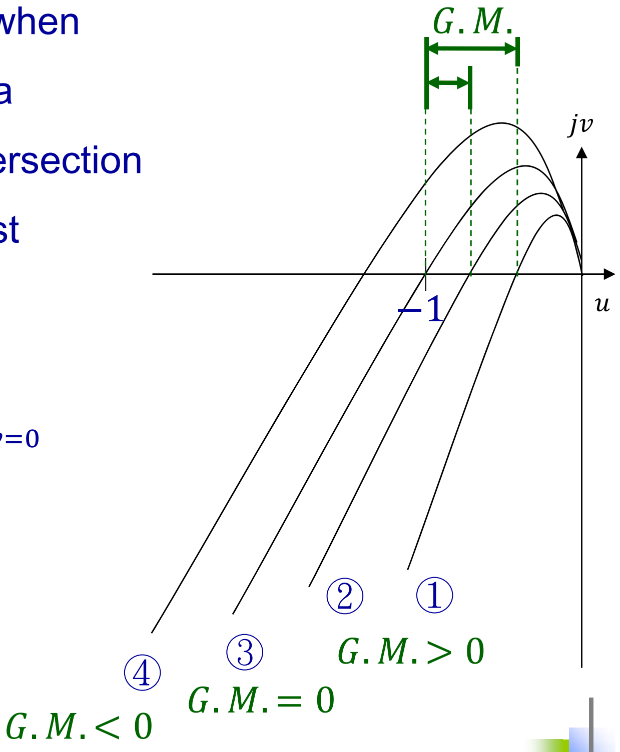
Gain Margin and Phase Margin -1

Gain margin

- The increase in the system gain when phase = $-180°$ that will result in a marginally stable system with intersection of the $-1 + j0$ point on the Nyquist diagram

$$G.M. \triangleq 20\,log|1| - 20\,log|L(\omega)|_{v=0}$$
$$= 20\,log\frac{1}{|L(\omega)|_{v=0}}\ \ dB$$

$$G.M. = 0 - G_{dB}|_{v=0}\ dB$$

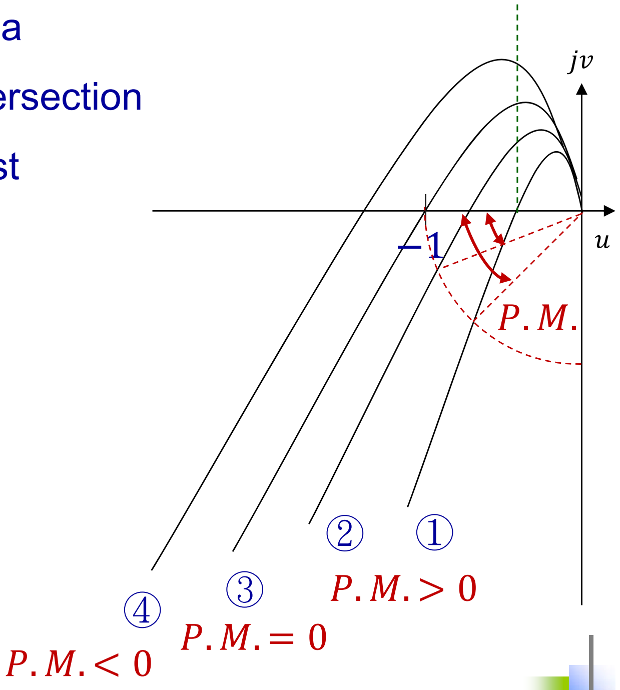


Gain Margin and Phase Margin -2

Phase margin

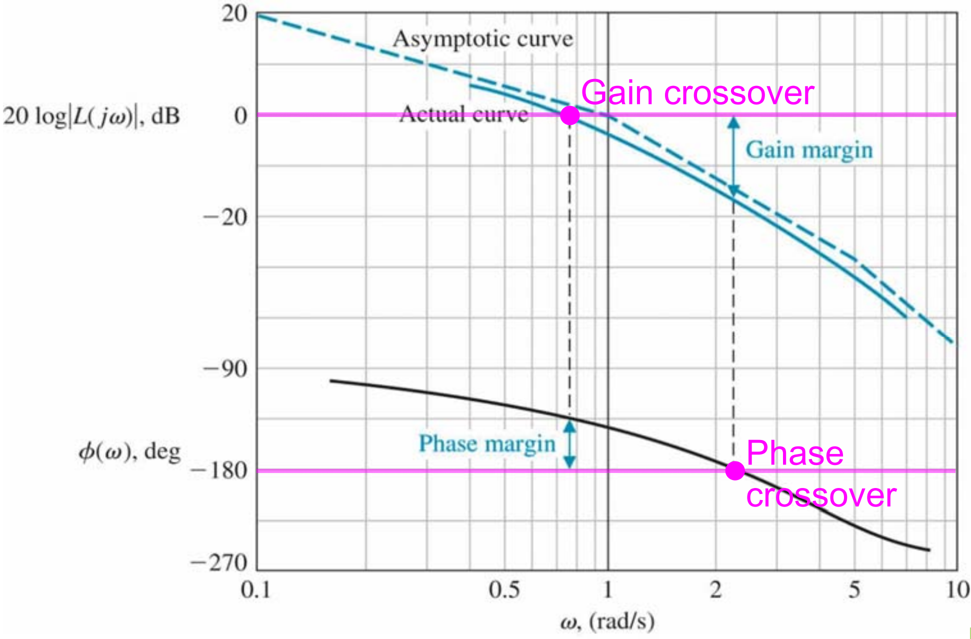
- The amount of phase shift of the $L(j\omega)$ at unity magnitude that will result in a marginally stable system with intersection of the $-1 + j0$ point on the Nyquist diagram

$$P.M. = \phi_{PM} = \angle L(\omega) - (-180^{0})$$



Gain Margin and Phase Margin -3

- Bode plot
  - Gain / phase crossover & gain / phase margin are more easily determined (on the Bode plot than on the Nyquist plot)

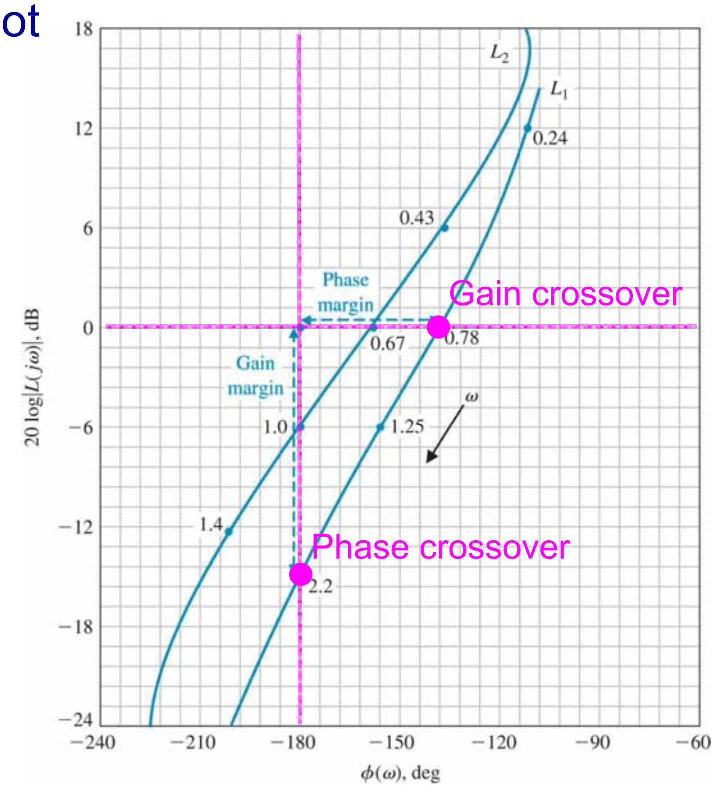


Copyright © 2011 Pearson Education, Inc. publishing as Prentice Hall

---

Gain Margin and Phase Margin -4

- Log-magnitude-phase plot



Copyright © 2011 Pearson Education, Inc. publishing as Prentice Hall

Gain Margin and Phase Margin -5

- Ex: A standard 2nd-order system

$$L(s) = GH(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

$$GH(j\omega) = \frac{\omega_n^2}{j\omega(j\omega+2\xi\omega_n)}$$

$$\left|GH(\omega_g)\right| = 1 = \frac{\omega_n^2}{\omega_g(\omega_g^2+4\xi^2\omega_n^2)^{\frac{1}{2}}}$$

Gain crossover

$$(\omega_g^2)^2 + 4\xi^2\omega_n^2(\omega_g^2) - \omega_n^4 = 0$$

$$\left(\frac{\omega_g^2}{\omega_n^2}\right)^2 + 4\xi^2\left(\frac{\omega_g^2}{\omega_n^2}\right) - 1 = 0$$

⇨ $$\frac{\omega_g^2}{\omega_n^2} = (4\xi^4+1)^{\frac{1}{2}} - 2\xi^2$$

Gain Margin and Phase Margin -6

$$P.M. = \phi_{PM} = -90^0 - \tan^{-1}\left(\frac{\omega_g}{2\xi\omega_n}\right) - (-180^0)$$

$$= \tan^{-1}(2\xi\left[\frac{1}{(4\xi^4+1)^{\frac{1}{2}}-2\xi^2}\right]^{\frac{1}{2}})$$

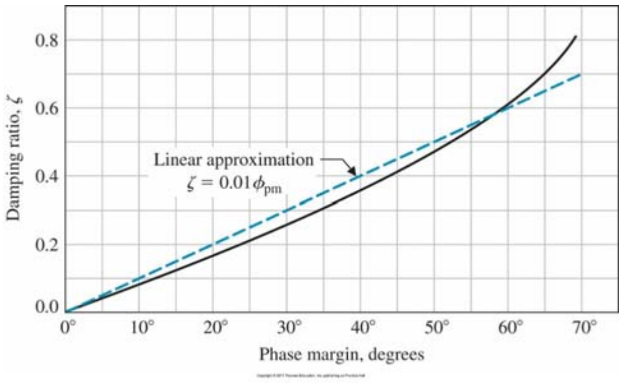
$$= \tan^{-1}\left(\frac{2}{\left[\left(4+\frac{1}{\xi^2}\right)^{\frac{1}{2}}-2\right]^{\frac{1}{2}}}\right)$$

$approximation$:

$\xi = 0.01\phi_{PM}$ $\quad$ $\xi \leq 0.7$

Adjust ϕ_{PM} in frequency response is EQUAL to adjust ξ in time response

$G.M. \rightarrow \infty$



The O.l. T.F. vs. C.l. T.F. -1

- Question: Can we obtain closed-loop frequency response from the open-loop frequency response?

Assuming unity feedback $H(j\omega) = 1$

$$G_c G(j\omega) = \mathrm{u} + \mathrm{j}\upsilon$$

Closed-loop T.F. $T(j\omega) = \frac{G_c G(j\omega)}{1+G_c G(j\omega)} = \frac{u+jv}{(1+u)+jv} = M(\omega)e^{j\emptyset(\omega)}$

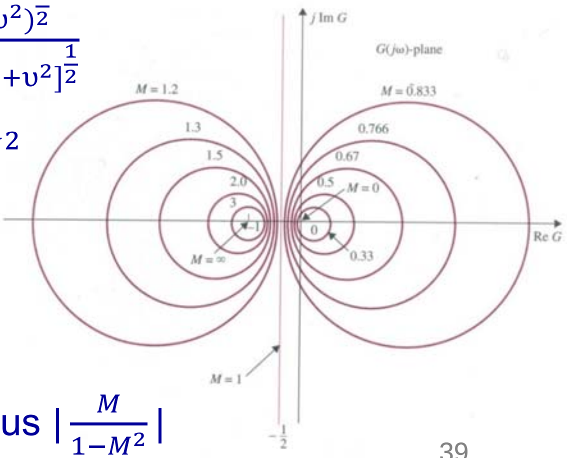
$$M(\omega) = \left|\frac{G_c G(j\omega)}{1+G_c G(j\omega)}\right| = \left|\frac{u+j\upsilon}{1+u+j\upsilon}\right| = \frac{(u^2+\upsilon^2)^{\frac{1}{2}}}{[(1+u)^2+\upsilon^2]^{\frac{1}{2}}}$$

$$(1-M^2)u^2 + (1-M^2)\upsilon^2 - 2M^2 u = M^2$$

$$u^2 + \upsilon^2 - \frac{2M^2}{1-M^2}u = \frac{M^2}{1-M^2}$$

$$\Rightarrow (u - \frac{M^2}{1-M^2})^2 + \upsilon^2 = (\frac{M}{1-M^2})^2$$

A circle: center at $(\frac{M^2}{1-M^2}, 0)$, radius $|\frac{M}{1-M^2}|$



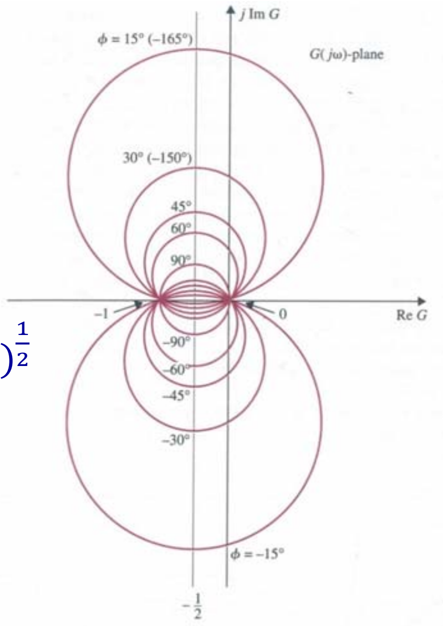
The O.l. T.F. vs. C.l. T.F. -2

$$\tan(\phi(\omega)) = N = \frac{v}{u+u^2+v^2}$$

$$u^2 + v^2 + u - \frac{v}{N} = 0$$

$$\left(u+\frac{1}{2}\right)^2 + \left(v-\frac{1}{2N}\right)^2 = \frac{1}{4}\left(1+\frac{1}{N^2}\right)$$

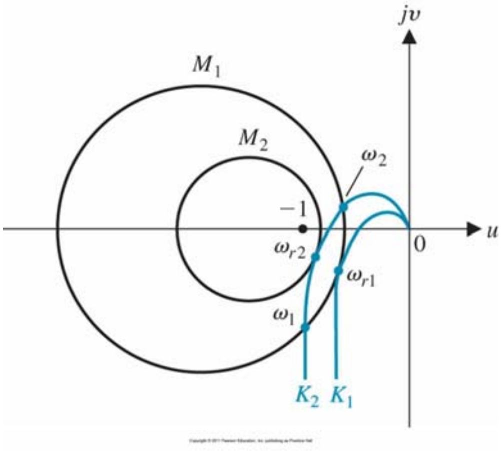
A circle: center at $\left(-\frac{1}{2}, \frac{1}{2N}\right)$, radius $\frac{1}{2}(1+\frac{1}{N^2})^{\frac{1}{2}}$

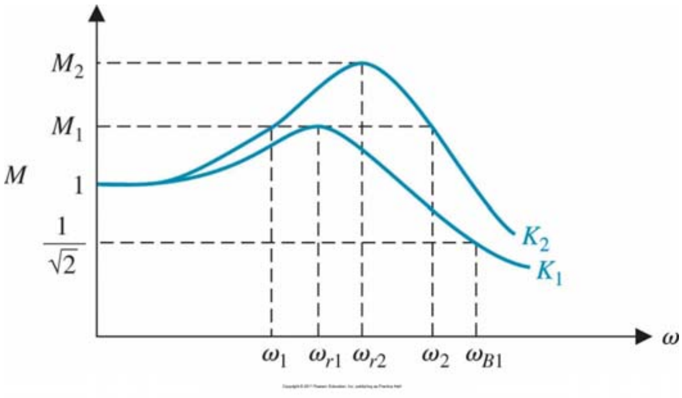


The O.l. T.F. vs. C.l. T.F. -3

- Ex: A system with two different gains, K_1 & K_2

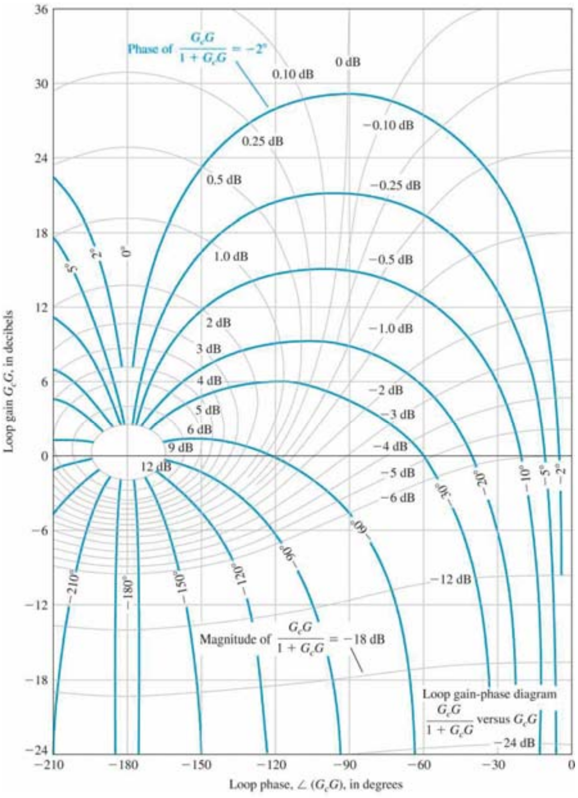
由open-loop T.F.的polar plot軌跡和M圓軌跡相交的狀態，可
以推估出此系統在closed-loop後的frequency response狀態





Nichols Chart -1

- Plotting magnitude and phase of the closed-loop system as contours on the log-magnitude-phase diagram



Nichols Chart -2

- Ex: $G(j\omega) = \frac{0.64}{j\omega[(j\omega)^2+j\omega+1]}$

  $H(j\omega) = 1$

  $\zeta = 0.5$

  $P.M. = 30°$

  C.I. $20\log M_{p\omega} = 9\ dB$

  at $\omega_r = 0.9$