Chap 5 Jacobians: Velocities and Static Forces

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Time-varying Position and Orientation -1

- Differentiation of a position vector P_Q

$$^{B}V_Q = \frac{d}{dt}\,{}^{B}P_Q = \lim_{\Delta t \to 0} \frac{{}^{B}P_Q(t+\Delta t) - {}^{B}P_Q(t)}{\Delta t}$$

Derivative of position vector $^{B}P_Q$ relative to frame $\{B\}$

$$^{A}({}^{B}V_Q) = {}^{A}(\frac{d}{dt}\,{}^{B}P_Q)$$

Expressed in frame $\{A\}$

$$= {}^{A}_{B}R\,{}^{B}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q$$

When both frames are the same

$$v_C = {}^{U}V_{C\,ORG}$$

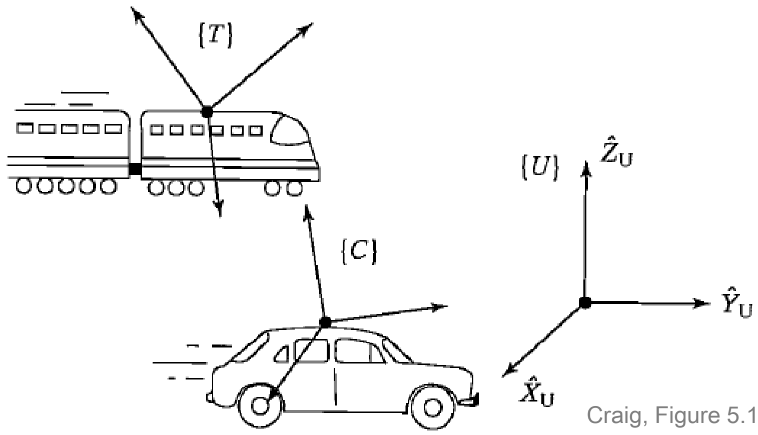
Velocity of the origin of frame $\{C\}$ relative to the universe reference frame $\{U\}$

Time-varying Position and Orientation -2

- Example

$^{U}V_{T} = 100\hat{\imath}$

$^{U}V_{C} = 30\hat{\imath}$



Craig, Figure 5.1

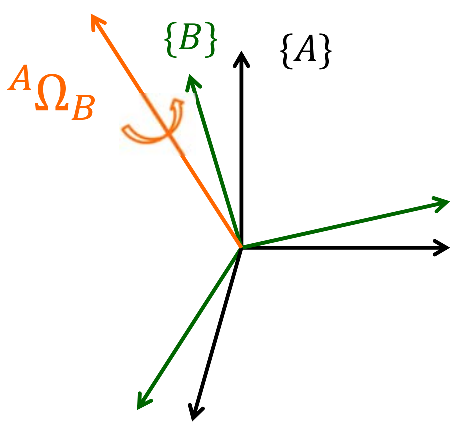
$$^{U}\left(\frac{d}{dt}\,^{U}P_{C\,ORG}\right) = \,^{U}V_{C\,ORG} = \upsilon_C = 30\hat{\imath}$$

$$^{C}(\,^{U}V_{T\,ORG}) = \,^{C}\upsilon_T = \,^{C}_{U}R(\upsilon_T) = \,^{C}_{U}R(100\hat{\imath}) = \,^{U}_{C}R^{-1}100\hat{\imath}$$

$$^{C}(\,^{T}V_{C\,ORG}) = \,^{C}_{T}R(\,^{T}(\,^{T}V_{C\,ORG})) = \,^{C}_{T}R(\,^{T}V_{C\,ORG})$$
$$= \,^{C}_{U}R\,^{U}_{T}R(-70\hat{\imath}) = -\,^{U}_{C}R^{-1}\,^{U}_{T}R70\hat{\imath}$$

Time-varying Position and Orientation -3

- Angular velocity vector $^{A}\Omega_{B}$
  - The rotation of frame $\{B\}$ relative to frame $\{A\}$
  - Direction of $^{A}\Omega_{B}$: The instantaneous axis of rotation
  - Magnitude of $^{A}\Omega_{B}$: The speed of rotation



$^{C}(\,^{A}\Omega_{B})$

Expressed in frame $\{C\}$

$\omega_c = \,^{U}\Omega_{C}$

Angular velocity of frame $\{C\}$ relative to the universe reference frame $\{U\}$

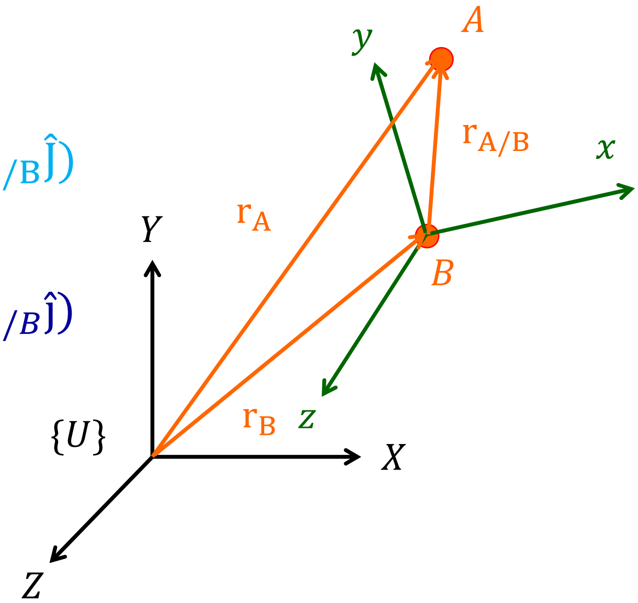
Rigid Body Motion -1

- Freshman Dynamics

$$\overrightarrow{r_A} = x_A\hat{I} + y_A\hat{J}$$
$$= \overrightarrow{r_B} + \overrightarrow{r_{A/B}}$$
$$= (x_B\hat{I} + y_B\hat{J}) + (x_{A/B}\hat{I} + y_{A/B}\hat{J})$$
$$= \overrightarrow{r_B} + \overrightarrow{r_{A/B}}$$
$$= (x_B\hat{I} + y_B\hat{J}) + (x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath})$$

↓ diff.

$$\overrightarrow{v_A} = \dot{\overrightarrow{r_A}} = \dot{x}_A\hat{I} + \dot{y}_A\hat{J}$$
$$= \dot{\overrightarrow{r_B}} + \dot{\overrightarrow{r_{A/B}}}$$
$$= (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{I} + \dot{y}_{A/B}\hat{J})$$

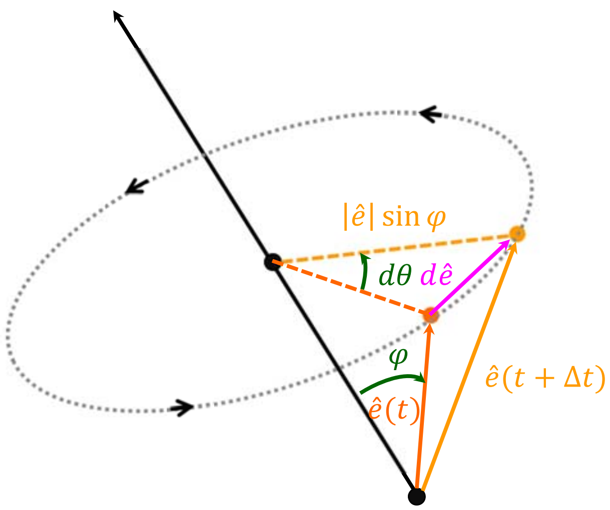


Rigid Body Motion -2

- $\overrightarrow{v_A} = \dot{\overrightarrow{r_B}} + \dot{\overrightarrow{r_{A/B}}}$

$$= (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \underline{(x_{A/B}\,\dot{\hat{\imath}} + y_{A/B}\dot{\hat{\jmath}})}$$
$$= x_{A/B}\,(\vec{\omega} \times \hat{\imath}) + y_{A/B}\,(\vec{\omega} \times \hat{\jmath})$$

$\vec{\omega}$ $\quad |\vec{\omega}| = \dot{\theta}$



Magnitude:

$$|d\hat{e}| = |\hat{e}|sin\varphi d\theta$$
$$|\dot{\hat{e}}| = |\hat{e}|sin\varphi\dot{\theta} = |\hat{e}||\vec{\omega}|sin\varphi$$

Direction:

$$d\hat{e} \perp \hat{e}$$
$$d\hat{e} \perp \vec{\omega}$$

⇨ $\dot{\hat{e}} = \vec{\omega} \times \hat{e}$

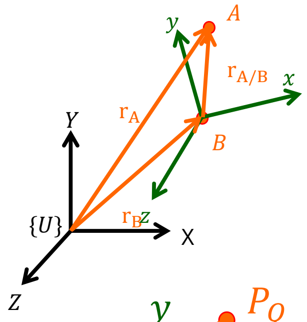
Rigid Body Motion -3

$$\overrightarrow{v_A} = (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega}\times(x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath})$$

$$= (\dot{x}_B\hat{I} + \dot{y}_B\hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega}\times(x_{A/B}\hat{I} + y_{A/B}\hat{J})$$

$$\Rightarrow \quad \overrightarrow{v_A} = \overrightarrow{v_B} + \overrightarrow{v_{rel}} + \vec{\omega}\times\overrightarrow{r_{A/B}}$$

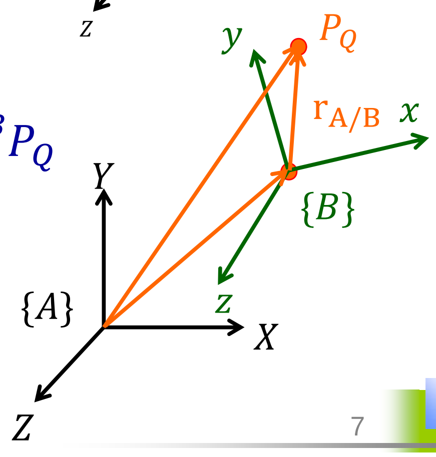
"relative" velocity



- Thus,

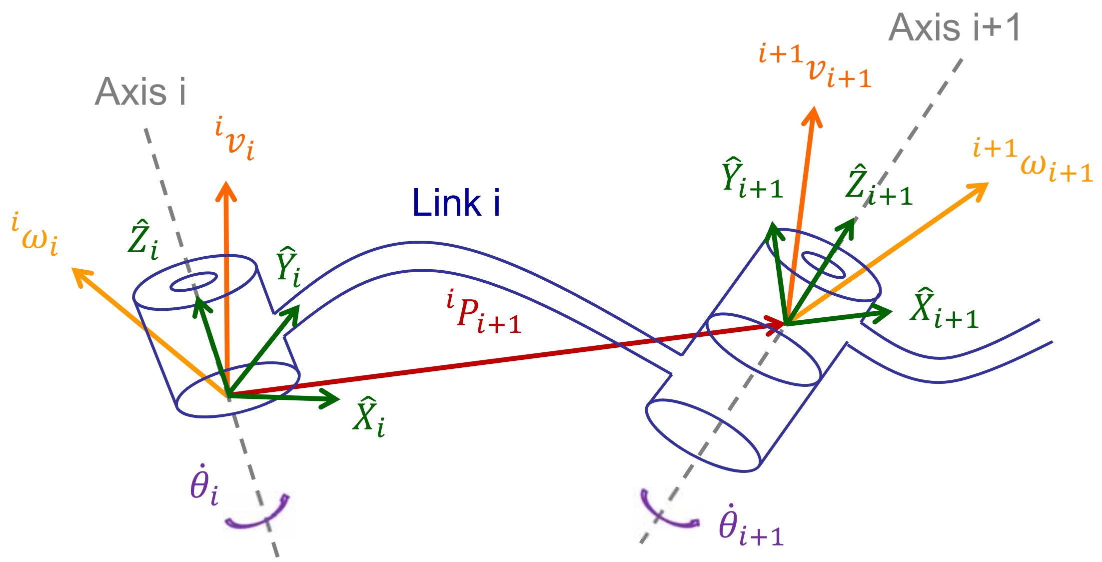
$$^{A}V_Q = {}^{A}V_{B\,ORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q$$

"relative" velocity



Velocity "Propagation" from Link to Link -1

- Strategy: Represent linear and angular velocities of link i in frame $\{i\}$, and find their relationship to those of neighboring links



Velocity "Propagation" from Link to Link -2

- Rotational Joint (Link i+1)
  - Angular velocity propagation

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_i + {}_{i+1}^{\;\;i}R\,\underline{\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}} $$

$$ \downarrow\ {}_{i}^{i+1}R \qquad\qquad \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

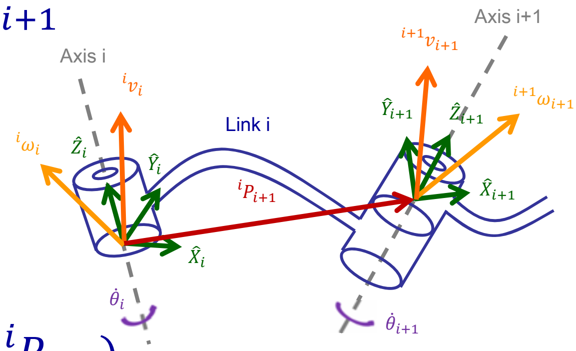
$$ {}^{i+1}\omega_{i+1} = {}_{i}^{i+1}R\ {}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

  - Linear velocity propagation

$$ {}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} $$

$$ \downarrow\ {}_{i}^{i+1}R $$

$$ {}^{i+1}v_{i+1} = {}_{i}^{i+1}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) $$



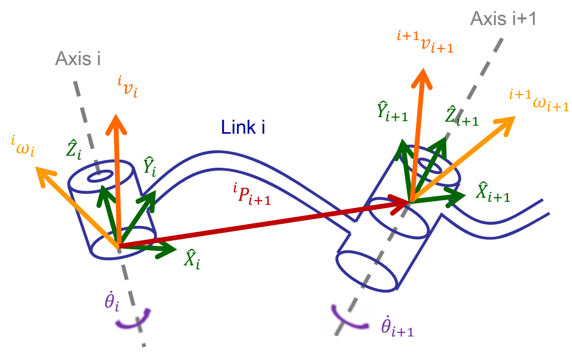
Velocity "Propagation" from Link to Link -3

- Prismatic joint (Link i+1)
  - Angular velocity propagation

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_i $$

$$ \downarrow\ {}_{i}^{i+1}R $$

$$ {}^{i+1}\omega_{i+1} = {}_{i}^{i+1}R\ {}^{i}\omega_i $$



  - Linear velocity propagation

$$ {}^{i}v_{i+1} = \left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}_{i+1}^{\;\;i}R\,\underline{\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}} $$

$$ \downarrow\ {}_{i}^{i+1}R \qquad\qquad \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} $$

$$ {}^{i+1}v_{i+1} = {}_{i}^{i+1}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

Jacobians -1

- A multidimensional form of the derivative

$$y_1 = f_1\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$y_2 = f_2\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$\vdots$$

$$y_6 = f_6\,(x_1, x_2, x_3, x_4, x_5, x_6)$$

⇒ $Y = F(X)$

Jacobians -2

- Calculating the differentials of y_i as a function of differentials of x_i

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6}\delta x_6$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6$$

Jacobian, "linear transformation"

⇒ $\delta Y = \frac{\partial F}{\partial X}\delta X = J(X)\delta X$

Function of X, if f_i is nonlinear

⇒ $\dot{Y} = J(X)\dot{X}$

Jacobians -3

- In robotics
  - Relating joint velocities to Cartesian velocities of the tip of the arm

$$^{0}\boldsymbol{\nu} = \begin{bmatrix} ^{0}v \\ ^{0}\omega \end{bmatrix} = {}^{0}J(\Theta)\dot{\Theta}$$

3x1 : plane motion

6x1 : spatial motion

- Changing a Jacobian's frame of reference (spatial motion)

$$^{B}\boldsymbol{\nu} = \begin{bmatrix} ^{B}v \\ ^{B}\omega \end{bmatrix} = {}^{B}J(\Theta)\dot{\Theta}$$

$$^{A}\boldsymbol{\nu} = \begin{bmatrix} ^{A}v \\ ^{A}\omega \end{bmatrix} = {}^{A}J(\Theta)\dot{\Theta} = \begin{bmatrix} ^{A}_{B}R & 0 \\ 0 & ^{A}_{B}R \end{bmatrix} \begin{bmatrix} ^{B}v \\ ^{B}\omega \end{bmatrix}$$

$$\Rightarrow \quad {}^{A}J(\Theta) = \begin{bmatrix} ^{A}_{B}R & 0 \\ 0 & ^{A}_{B}R \end{bmatrix} {}^{B}J(\Theta)$$

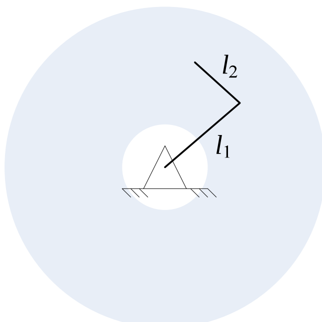
Jacobians -4

- Invertibility

$$\dot{\Theta} = J^{-1}(\Theta)\boldsymbol{v}$$

  - Singular: When the Jacobian J is NOT invertible
    - Workspace-boundary singularities

      Ex: When the manipulator is fully stretch out or folded back on itself
    - Workspace-interior singularities
  - When a manipulator is in a singular configuration
    - Lost one or more DOF



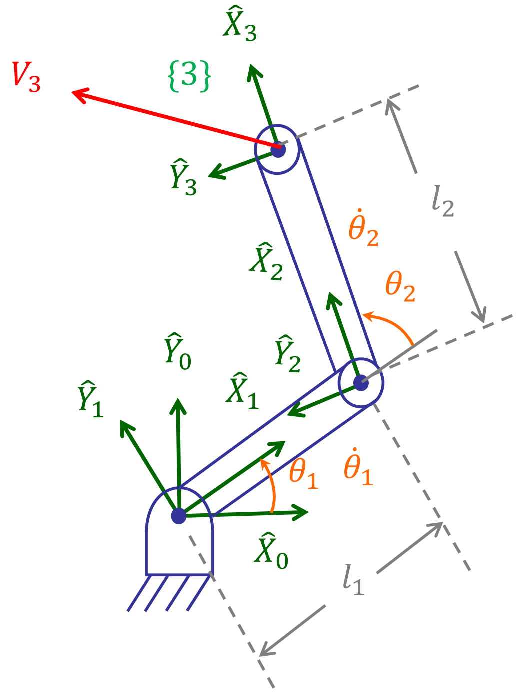
Example: A RR Manipulator -1

- Method 1: Velocity "propagation" from link to link

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Example: A RR Manipulator -2

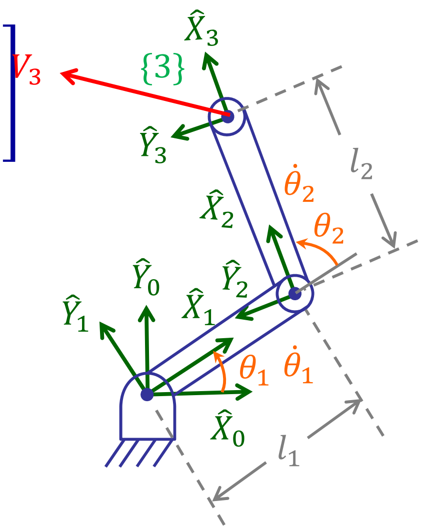
- Link "propagation"

$$ {}^1\omega_1 = {}^1_0R\ {}^0\omega_0 + \dot{\theta}_1\ {}^1\hat{Z}_1 = \dot{\theta}_1\ {}^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} $$

$$ {}^1v_1 = {}^1_0R({}^0v_0 + {}^0\omega_0 \times {}^0P_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^2\omega_2 = {}^2_1R\ {}^1\omega_1 + \dot{\theta}_2\ {}^2\hat{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} $$

$$ {}^2v_2 = {}^2_1R({}^1v_1 + {}^1\omega_1 \times {}^1P_2) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} $$



Example: A RR Manipulator -3

$$ {}^3\omega_3 = {}^2\omega_2 $$

$$ {}^3v_3 = {}^3_2R\left({}^2v_2 + {}^2\omega_2 \times {}^2P_3\right) $$

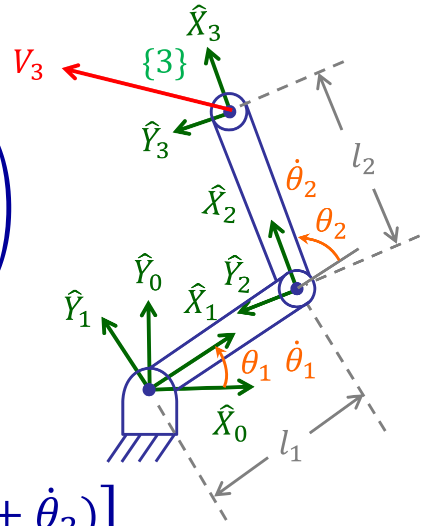
$$ = I\left(\begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}\right) $$

$$ = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} $$

$$ {}^0v_3 = \underline{{}^0_3R}\ {}^3v_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} $$

$$ = {}^0_1R\,{}^1_2R\,{}^2_3R $$

$$ = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



Example: A RR Manipulator -4

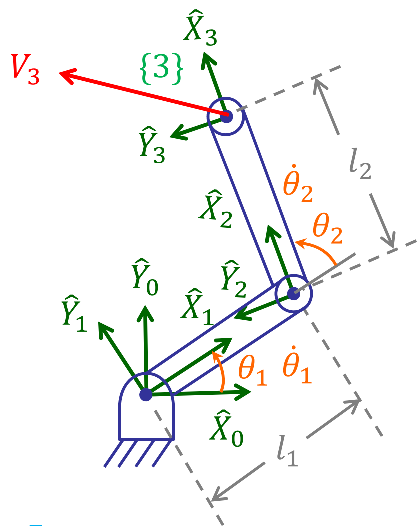
- Therefore

$$ {}^3\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

$$ = {}^3J(\Theta)\dot{\Theta} $$

$$ det\begin{vmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{vmatrix} = l_1 l_2 s_2 = 0 $$

⇨ $\theta_2 = 0\ or\ 180$

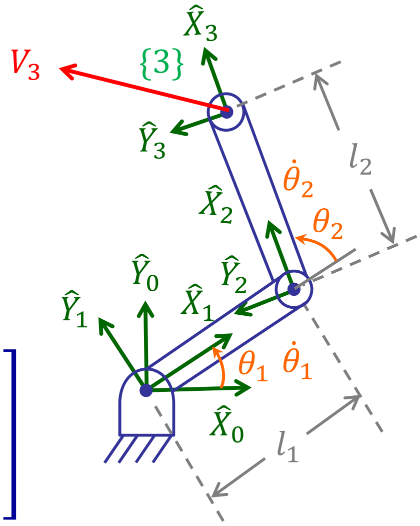


$$ {}^0\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

$$ = {}^0J(\Theta)\dot{\Theta} $$

Example: A RR Manipulator -5

- Method 2: Direct differentiation



$$^0\begin{bmatrix} p_x \\ p_y \\ \theta \end{bmatrix} = {}^0\begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ \theta_1 + \theta_2 \end{bmatrix}$$

↓ diff.

$$^0\begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = {}^0\begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$$= {}^0\begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\dot{X} = {}^0J(\Theta)\dot{\Theta}$$

Note: NO 3x1 orientation vector whose derivative is ω

Static Forces in Manipulators -1

- When considering static forces
  - Lock all the joints
  - Write force-moment relationship
  - Compute static torque (ignore gravity)

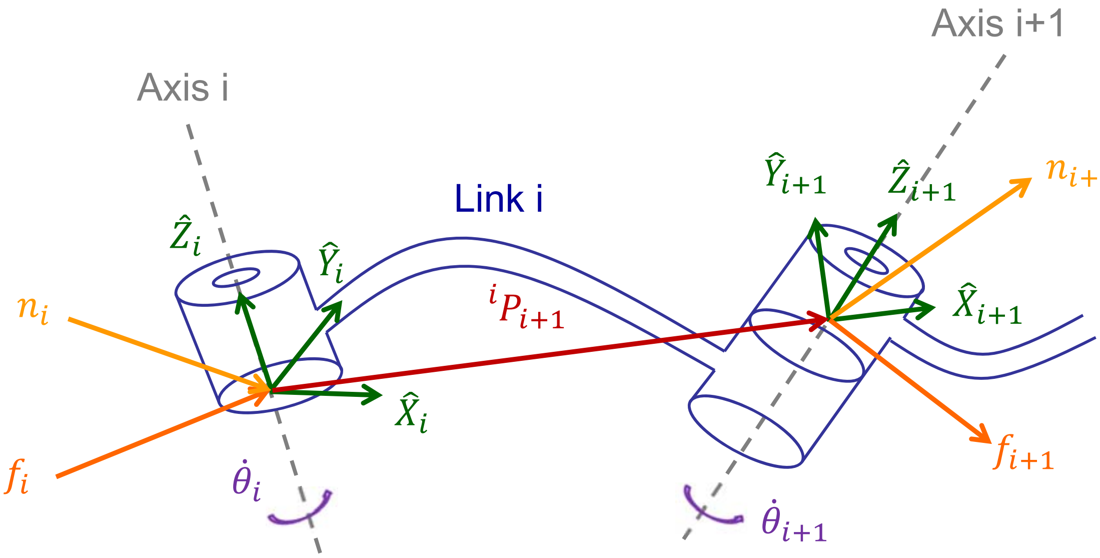
Static Forces in Manipulators -2

- $^{i}f_i = {}^{i}f_{i+1}$ $\quad\Big\downarrow {}^{i}_{i+1}R\quad$ $^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$

$$^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1}$$

$$\Big\downarrow {}^{i}_{i+1}R$$

$$^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i$$



Static Forces in Manipulators -3

- The joint toque required to maintain the static equilibrium
  - Revolute joint

$$\tau_i = {}^{i}n_i^T\ {}^{i}\hat{Z}_i$$

  - Prismatic joint

$$\tau_i = {}^{i}f_i^T\ {}^{i}\hat{Z}_i$$

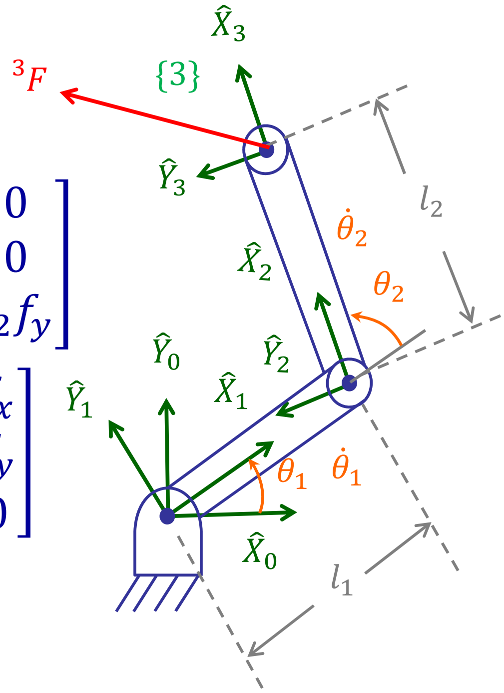
Example: A RR Manipulator -1

- Force "propagation" from link to link

$$ {}^{2}f_2 = {}^{2}_{3}R\ {}^{3}f_3 = I\ {}^{3}F = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} $$

$$ {}^{2}n_2 = {}^{2}_{3}R\ {}^{3}n_3 + {}^{2}P_3 \times {}^{2}f_2 = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} $$

$$ {}^{1}f_1 = {}^{1}_{2}R\ {}^{2}f_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix} $$



Example: A RR Manipulator -2

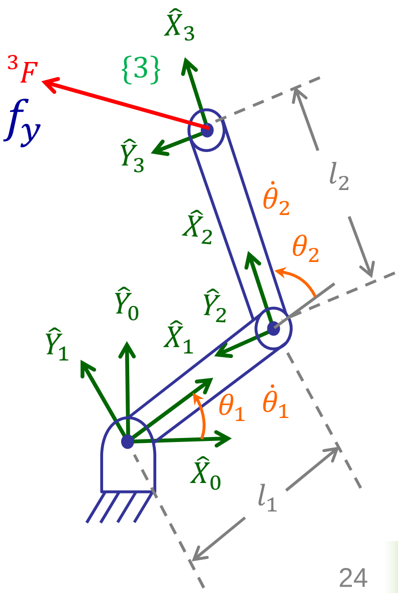
$$ {}^{1}n_1 = {}^{1}_{2}R\ {}^{2}n_2 + {}^{1}P_2 \times {}^{1}f_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix} $$

- Therefore,

$$ \tau_1 = {}^{1}n_1^T\ {}^{1}\widehat{Z_1} = l_1 s_2 f_x + (l_1 c_2 + l_2) f_y $$

$$ \tau_2 = {}^{2}n_2^T\ {}^{2}\widehat{Z_2} = l_2 f_y $$

$$ \Rightarrow \tau = \begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} $$



Jacobian in the Force Domain

- The principal of virtual work

$$F \cdot \delta\mathcal{X} = \Gamma \cdot \delta\Theta$$

$$F^T\delta\mathcal{X} = F^T J\delta\Theta = \Gamma^T\delta\Theta$$

$$\Gamma = J^T F$$

Respect to frame $\{0\}$

$$\Rightarrow \quad \Gamma = {}^0J^T\ {}^0F$$

"inverse" Cartesian torque to joint torque without using IK technique

Cartesian Transformation -1

- General velocity and force representations

$$\boldsymbol{\nu} = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad \boldsymbol{\mathcal{F}} = \begin{bmatrix} F \\ N \end{bmatrix}$$

- Frame transformation

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i + \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R(\ {}^{i}v_i + \ {}^{i}\omega_i \times \ {}^{i}P_{i+1})$$

$$\downarrow i = A, i+1 = B,\ \dot{\theta} = 0$$

$$\begin{bmatrix} {}^{A}v_A \\ {}^{A}\omega_A \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B\,ORG} \times {}^{A}_{B}R \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}v_B \\ {}^{B}\omega_B \end{bmatrix}$$

$${}^{A}\boldsymbol{\nu}_A = {}^{A}_{B}T_v\ {}^{B}\boldsymbol{\nu}_B \qquad P \times = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}$$

Cartesian Transformation -2

↓ "inverse"

$$\begin{bmatrix} {}^{B}v_B \\ {}^{B}\omega_B \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & -{}^{B}_{A}R\,{}^{A}P_{BORG}\times \\ 0 & {}^{B}_{A}R \end{bmatrix} \begin{bmatrix} {}^{A}v_A \\ {}^{A}\omega_A \end{bmatrix}$$

$${}^{B}\boldsymbol{v}_B = {}^{B}_{A}T_v\,{}^{A}\boldsymbol{v}_A$$

- Similarly,

$$\begin{bmatrix} {}^{A}F_A \\ {}^{A}N_A \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ {}^{A}P_{BORG}\times {}^{A}_{B}R & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}F_B \\ {}^{B}N_B \end{bmatrix}$$

$${}^{A}\boldsymbol{\mathcal{F}}_A = {}^{A}_{B}T_f\,{}^{B}\boldsymbol{\mathcal{F}}_B$$

⇒ $${}^{A}_{B}T_f = {}^{A}_{B}T_v{}^{T}$$

The End

- Questions?